

# $K$ -TWISTED EQUIVARIANT $K$ -THEORY FOR $SU(N)$

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## Abstract

We present a version of twisted equivariant  $K$ -theory- $K$ -twisted equivariant  $K$ -theory, and use Grothendieck differentials to compute the  $K$ -twisted equivariant  $K$ -theory of simple simply connected Lie groups. We did the calculation explicitly for  $SU(N)$  explicitly. The basic idea is to interpret an equivariant gerbe as an element of equivariant  $K$ -theory of degree 1.

## 1 Introduction

Let  $G$  be a finite dimensional simple Lie group, a classical question related to it is to understand the space  $Hom(\pi, G)/G$ , where  $\pi$  is a finitely presented group. This space  $Hom(\pi, G)/G$  is the moduli space of flat connections on a principal  $G$ -bundle on a manifold with fundamental group  $\pi$ . Because of Atiyah and Segal's result [2], and the fact that  $K$ -theory is defined for a large class of geometric objects including usual topological spaces and non-commutative ones, our first approach is to study the equivariant  $K$ -theory of  $Hom(\pi, G)$ . We get the answer for the case  $\pi = \mathbb{Z}$ , i.e. the equivariant  $K$ -theory  $K_G^*(G) \cong \Omega_{R(G)/\mathbb{Z}}^*$  [8] the algebra

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of Grothendieck differentials of the representation ring  $R(G)$  of  $G$  over  $\mathbb{Z}$  when  $G$  is compact and the fundamental group is torsion free (the general situation is still open). This is the origin of our project about Grothendieck differentials in  $K$ -theory.

We get interested in twisted  $K$ -theory because of Freed-Hopkins-Teleman's result on twisted equivariant  $K$ -theory and Verlinde algebras [9], [10], [11], that is for a Lie group  $G$ , the Verlinde algebra  $V_k(G)$  at level  $k$  is twisted equivariant  $K$ -theory of  $G$  (with adjoint action) at particular degree. Unfortunately, they didn't publish their proof yet. The main idea for this paper is to use Grothendieck differentials to give a partial proof of their result, and supply a candidate for the geometric definition of twisted  $K$ -theory.

The first question we need solve is to find a good geometric model for twisted equivariant  $K$ -theory. Let  $\mathbb{H}$  be a infinite dimensional separable Hilbert space, and  $U = U(\mathbb{H})$  be the set of unitary operators on  $\mathbb{H}$ , we know that  $U$  is contractible. The group  $U$  has a natural subgroup  $\{e^{i\theta}I\}$  which is isomorphic to  $S^1$ , let us denote the quotient group by  $PU$ . For a topological space  $X$ , in principle, a twistor is a principal  $PU$ -bundle over  $X$ , thus an element in  $H^3(X, \mathbb{Z})$ . Naturally a geometric realization of  $H^3(X, \mathbb{Z})$  elements is needed. We already have a geometric realization of  $H^3$  classes, i.e., gerbes [6]. Based on the idea of gerbes, there are some other geometric realizations, like bundle gerbes [17] or central extensions of groupoids [5] [19]. All these involves infinite dimensional objects. We are more interested in finite dimensional realization of gerbes, like [15] [12]. But how can we do twisting with gerbes? As far as I know, there is no clean geometric definition for twisted  $K$ -theory. The equivariant situation is more subtle, in this case, whether to use equivariant gerbes [7] to do twisting is questionable.

We present a solution to these questions in nice situation. We study the twisted equivariant  $K$ -theory of  $G$  (with adjoint action). In this case, we interpret an equivariant gerbe as an element of  $K^1$ , then based on this  $K^1$  element, we give an intuitive definition of  $K$ -twisted equivariant  $K$ -theory. The paper is basically two parts. In the first part, we prove that an element in the equivariant cohomology  $H_G^3(G)$  can be interpreted as an element of  $K_G^1(G)$ , and in the second part, we use the definition we give to do calculation for  $SU(N)$  explicitly (in fact we did the calculation for classical groups, but for simplicity and to demonstrate the idea, we just present the case for  $SU(N)$ ).

## 2 $K$ -Twisted $K$ -theory

In the section, we first discuss the general picture of twisted  $K$ -theory and then present our definition for  $K$ -twisted (equivariant)  $K$ -theory.

$K$ -theory is a generalized cohomology theory [1]. For a paracompact topological space  $X$ ,  $K^*(X)$  has several equivalent definitions:

1. Geometric definition: equivalence classes of complex of vector bundles over  $X$ .

2. Homotopic definition: Homotopy classes of maps:  $[X, Fred]$ ,  $[X, Fred_{as}]$ , where  $Fred$  and  $Fred_{as}$  are the set of Fredholm operators and self-adjoint operators in  $\mathbb{H}$ .

3. Algebraic definition:  $K$ -theory of  $C^*$ -algebra  $C_0(X)$ .

Based on the homotopic definition of  $K$ -theory, the general picture of the twisted  $K$ -theory can be as follows. If we have a principal  $PU$ -bundle  $P$  over  $X$ , notice there are natural actions of  $PU$  on  $Fred$  and  $Fred_{as}$ , we can form the spaces  $P \times_{PU} Fred = (P \times Fred)/PU$  and  $P \times_{PU} Fred_{as}$ , which are fiber bundles over  $X$ , then we can define the twisted  $K$ -theory as the homotopy classes of sections of these two bundles. There are general definitions of twisted  $K$ -theory from point of view of  $C^*$ -algebra, see [16], or [19] for the equivariant cases for detail. We are more interested in a geometric picture of twisted  $K$ -theory, and if possible, a definition with finite dimensional objects.

The twistor, i.e., the principal  $PU$ -bundle over  $X$  is classified by  $H^1(X, PU)$ . The exact sequence of groups  $1 \rightarrow S^1 \rightarrow U \rightarrow PU \rightarrow 1$  implies that  $PU$  is a model for  $BS^1$ , the classifying space of  $S^1$ . Thus  $H^1(X, PU) \cong H^2(X, S^1) \cong H^3(X, \mathbb{Z})$ . So the twistor is classified by  $H^3(X, \mathbb{Z})$ . The geometric construction of a class in  $H^3(X, \mathbb{Z})$  is a gerbe [6]. In brief, we use a gerbe to do twisted  $K$ -theory.

One might hope to use vector bundles to construct the twisted  $K$ -theory geometrically. This is succeeded only in case that the twistor is a torsion element in  $H^3(X, \mathbb{Z})$  [4]. In this case the twisted  $K$ -theory is the Grothendieck group of the category of twisted bundles. The essential problem is the non-existence of finite dimensional twisted bundles in general.

The geometric picture for the twisted equivariant  $K$ -theory is more subtle. Let  $G$  be a topological group,  $X$  be a  $G$ -space, the equivariant  $K$ -theory  $K_G^*(X)$  can be defined in the similar ways [18]. The question in this case is what kind of twistor we can use. The natural generalization of non-equivariant case is the elements in  $H_G^3(X)$ , the 3rd degree equivariant cohomology, in other words equivariant gerbes. But there is some problem if we use it to a geometric approach. The reason is that an element of  $H_G^3(X)$  is an object on  $EG \times_G X$ , not exactly an equivariant object on

$X$ . There is a question just like non-equivariant case, what kind geometric objects we can use, again the non-existence of twisted equivariant bundle is a problem.

There is another point of view for the whole picture. Let  $X$  be a finite dimensional object, for example, a finite dimensional manifold, then the chern character  $ch : K^1(X) \otimes \mathbb{Q} \cong H^{odd}(X, \mathbb{Q})$ . So up to  $\mathbb{Z}$ -torsion, an element in  $H^3(X, \mathbb{Z})$  can be viewed as an element in  $K^1(X)$ . This simple observation suggests the following intuitive definition of  $K$ -twisted  $K$ -theory.

**DEFINITION 2.1** *Let  $X$  be a topological space, and  $\alpha \in K^1(X)$ , the  $K$ -twisted  $K$ -theory  ${}^\alpha K^*(X)$  is the homology of the following complex,*

$$\cdots \xrightarrow{\wedge^\alpha} K^0(X) \xrightarrow{\wedge^\alpha} K^1(X) \xrightarrow{\wedge^\alpha} K^0(X) \xrightarrow{\wedge^\alpha} \cdots$$

The desired properties of twisted  $K$ -theory are obvious from this Definition. This definition should agree with the homotopic definition in case  $\alpha$  is a non-torsion element in  $H^3(X, \mathbb{Z})$ , and there should be a more general geometric definition of twisted  $K$ -theory which generalizes this definition and twisted bundle in the torsion case. We are working on this topic.

This definition can be easily generalized to the equivariant case,

**DEFINITION 2.2** *Let  $X$  be a topological space,  $G$  be a compact topological group acting on  $X$ , and  $\alpha \in K_G^1(X)$ , the  $K$ -twisted  $K$ -theory  ${}^\alpha K_G^*(X)$  is the homology of the following complex,*

$$\cdots \xrightarrow{\wedge^\alpha} K_G^0(X) \xrightarrow{\wedge^\alpha} K_G^1(X) \xrightarrow{\wedge^\alpha} K_G^0(X) \xrightarrow{\wedge^\alpha} \cdots$$

### 3 The basic gerbe as an element of $K_G^1(G)$

Let  $G$  be a  $n$ -dimensional compact simple simply-connected Lie group of rank  $d$ ,  $T$  be a maximal torus of  $G$ , and  $W$  be the Weyl group of  $G$  with respect to  $T$ . We use  $R(G)$ ,  $R(T)$  to denote the representation rings of  $G$  and  $T$  respectively. If  $\chi_1, \chi_2, \dots, \chi_d$  are the simple characters of  $T$ , then the character group  $X^*(T) = \text{Hom}(T, S^1)$  is the free abelian group generated by  $\chi_1, \chi_2, \dots, \chi_d$ , and the representation ring  $R(T)$  is the group ring  $\mathbb{Z}[X^*(T)] = \mathbb{Z}[\chi_1, \chi_2, \dots, \chi_d, \chi_1^{-1}, \chi_2^{-1}, \dots, \chi_d^{-1}]$ . The Weyl group  $W$  acts on  $R(T)$ , the invariant subalgebra  $R(T)^W$  is the representation ring  $R(G)$ , which is a polynomial ring generated by “basic” representations  $\rho_1, \rho_2, \dots, \rho_d$  corresponding to a choice of a set of simple roots.

The cohomology of  $T$  can be easily described in terms of these characters. The character  $\chi_i : T \rightarrow S^1$  can be viewed as an element of  $[X, S^1] \cong H^0(X, S^1) \cong H^1(X, \mathbb{Z})$ , let us denote this element by  $\eta_i$ . By this way, we get a homomorphism of abelian groups  $X^*(T) \rightarrow H^1(T, \mathbb{Z})$ , and  $H^*(T, \mathbb{Z}) \cong \bigwedge(\eta_1, \eta_2, \dots, \eta_d)$ .

The  $K$ -theory can be described in similar way. A character  $\chi_i : T \rightarrow S^1 = U(1)$  defines a line bundle over the suspension of  $T$ , thus defines an element of  $K^1(T)$ , again we denote this element by  $\eta_i$ . Therefore we have a homomorphism between abelian groups:  $X^*(T) \rightarrow K^1(T)$ , and  $K^*(T) \cong \bigwedge(\eta_1, \eta_2, \dots, \eta_d)$ . In particular we see that there is an isomorphism  $c : K^*(T) \cong H^*(T, \mathbb{Z})$ , where the map is in fact the first chern class of bundles, and this map is equivariant under the action of Weyl group  $W$ .

Let  $X$  be a paracompact space,  $H$  be a compact topological group acting on  $X$ , then the equivariant cohomology is defined as  $H_H^*(X) = H^*(EH \times_H X)$  [3], where  $EH \rightarrow BH$  is a universal principal  $H$ -bundle,  $BH$  is a classifying space for  $H$ . In particular,  $H_H^*(pt) = H^*(BH)$ , and the bundle map  $EH \times_H X \rightarrow BH$  give  $H_H^*(X)$  a  $H_H^*(pt)$ -module structure.

In the case of the torus  $T$ , the coefficient ring  $H^*(BT)$  can also be described in terms of the character group  $X^*(T)$ . For any character  $\chi : T \rightarrow S^1$ , it defines a line bundle  $ET \times_T \mathbb{C}\chi$  over  $BT$ , the first chern class of this bundle gives an abelian group homomorphism:  $X^*(T) \rightarrow H^2(BT)$ , this induces an isomorphism between  $H^*(BT)$  and the symmetric algebra  $S_T$  of  $X^*(T)$ . Notice that  $H^*(BT)$  carries a natural action of the Weyl group  $W$ .

Let us consider  $G$  as a  $G$ -space with adjoint action, it is well-known that  $H_G^3(G, \mathbb{Z}) \cong \mathbb{Z}$ , and the generator (up to sign) is called the basic (equivariant) gerbe. There are several ways to describe this gerbe [5] [15] [12], the main result of this section is to present another way to view this basic gerbe.

Let us recall two lemmas about equivariant  $K$ -theory and equivariant cohomology of  $G$  [7] [8].

**LEMMA 3.1** *For a compact simple simply-connected Lie group  $G$ ,*

$$H_G^*(G) \cong (H^*(BT) \otimes H^*(T))^W$$

**LEMMA 3.2** *For a compact simple simply-connected Lie group  $G$ ,*

$$K_G^*(G) \cong (R(T) \otimes K^*(T))^W$$

**PROPOSITION 3.3** *For a compact simple simply-connected Lie group  $G$ , the basic equivariant gerbe can be viewed as an element of  $K_G^1(G)$ .*

*Proof.* By above lemmas,

$$\begin{aligned} H_G^3(G) &\cong (H^0(BT) \otimes H^3(T) \oplus H^2(BT) \otimes H^1(T))^W \\ &\subset (R(T) \otimes K^1(T))^W \cong K_G^1(G), \end{aligned}$$

here,  $H^0(BT) \cong \mathbb{Z}$ ,  $H^2(BT) \cong X^*(T)$  can be viewed as subset of  $R(T)$ .  $\square$

## 4 $K$ -Twisted $K$ -theory for $SU(N)$

In this section, we will use our definition of  $K$ -twisted equivariant  $K$ -theory and Grothendieck differentials to do the calculation for  $SU(N)$ .

Let us first recall some background of Grothendieck differentials. Let  $A \subset B$  be commutative rings. The algebra of Grothendieck differentials  $\Omega_{B/A}^*$  [13] is the differential graded  $A$ -algebra constructed as follows:

Let  $F$  be the free  $B$ -module generated by all elements in  $B$ , to be clear, we use  $db$  to denote the generator corresponding to  $b \in B$ , so

$$F = \bigoplus_{b \in B} Bdb.$$

and let  $I \subset F$  be the  $B$ -submodule generated by

$$\left\{ \begin{array}{l} da, \forall a \in A \\ d(b_1 + b_2) - db_1 - db_2, \forall b_1, b_2 \in B \\ d(b_1 b_2) - b_1 db_2 - b_2 db_1, \forall b_1, b_2 \in B \end{array} \right\},$$

we then get the quotient  $B$ -module

$$\Omega_{B/A} = F/I.$$

Let  $\Omega_{B/A}^0 = B$ ,  $\Omega_{B/A}^1 = \Omega_{B/A}$ , and  $\Omega_{B/A}^p = \Lambda_B^p \Omega_{B/A}$ . There is a differential:  $d : \Omega_{B/A}^p \rightarrow \Omega_{B/A}^{p+1}$ , which maps  $b \in B$  to  $db$ , then

$$\Omega_{B/A}^* = \bigoplus_{p=0}^{\infty} \Omega_{B/A}^p$$

is the differential graded algebra of Grothendieck differentials of  $B$  over  $A$ . It is the generalization of the algebra of differentials on affine spaces, for example, if  $B = A[x_1, \dots, x_n]$ , then  $\Omega_{A[x_1, \dots, x_n]/A}^p = \bigoplus_{i_1 < i_2 < \dots < i_p} A[x_1, \dots, x_n] dx_{i_1} \wedge \dots \wedge dx_{i_p}$ .

For any representation  $\rho : G \rightarrow GL(V)$ , it defines a vector bundle over the suspension of  $G$ , which is  $G$ -equivariant, so it defines an element  $d\rho$  of  $K_G^1(G)$ . The main result in [8] is this defines an isomorphism  $\Omega_{R(G)/\mathbb{Z}}^* \cong K_G^*(G)$ , when  $\pi_1(G)$  is torsion free.

This result applies to the case of a torus  $T$ . In terms of Grothendieck differentials, for any character  $\chi_i$  of  $T$ ,  $d\chi_i = \chi_i \eta_i$ , where  $\eta_i$  is the  $K$ -theory element or cohomology element of  $T$  constructed in the previous section, or in other words,  $\eta_i = \frac{d\chi_i}{\chi_i}$ .

In the case  $G = SU(N)$ , if we let  $\rho_i$  be the  $i$ -th elementary symmetric polynomial in  $\chi_1, \chi_2, \dots, \chi_N$ , then  $R(G) = \mathbb{Z}[\rho_1, \rho_2, \dots, \rho_{N-1}]$ , and the equivariant  $K$ -theory is  $K_G^*(G) = \wedge_{R(G)}(d\rho_1, d\rho_2, \dots, d\rho_{N-1})$ .

**PROPOSITION 4.1** *For  $SU(N)$ , let  $\delta$  be the basic gerbe, than as an element of  $K_{SU(N)}^1(SU(N))$ , is*

$$\begin{aligned}\delta &= \sum \chi_i \eta_i \\ n\delta &= \sum \chi_i^n \eta_i\end{aligned}$$

Let  $\alpha = n\delta$ , now we are going to calculate the  $K$ -twisted  $K$ -theory  ${}^\alpha K_G^*(G)$  for  $G = SU(N)$ , we need a lemma.

**LEMMA 4.2** *Let  $\alpha = \sum x_i^n dx_i$   $n \geq 0$ , then the following complex is exact except at the last spot:*

$$\begin{aligned}0 \rightarrow \mathbb{Z}[x_1, x_2, \dots, x_N] \xrightarrow{\wedge^\alpha} \oplus \mathbb{Z}[x_1, x_2, \dots, x_N] dx_i \xrightarrow{\wedge^\alpha} \\ \dots \xrightarrow{\wedge^\alpha} \mathbb{Z}[x_1, x_2, \dots, x_N] dx_1 \dots dx_N \xrightarrow{\wedge^\alpha} 0\end{aligned}$$

Now it is a standard calculation to get  $K$ -twisted equivariant  $K$ -theory, in particular,

**THEOREM 4.3** *Let  $\alpha = (N+k)\delta$ , then  ${}^\alpha K_{SU(N)}^N(SU(N))$  is the Verlinde algebra  $V_k$  of  $SU(N)$  at level  $k$ .*

*Proof* By the above lemma and taking  $W$ -invariants, the non-trivial term of  ${}^\alpha K_{SU(N)}^*(SU(N))$  only appears in degree  $N$ . If  $\alpha = a_i d\rho_i$  (These  $a_i \in R(SU(N))$  are classical functions, for example see [14]), then the  $K$ -twisted equivariant  $K$ -theory is  ${}^\alpha K_{SU(N)}^N(SU(N)) = R(SU(N)) d\rho_1 d\rho_2 \dots d\rho_{N-1} / (a_1, a_2, \dots, a_{N-1}) d\rho_1 d\rho_2 \dots d\rho_{N-1} \cong R(SU(N)) / (a_1, a_2, \dots, a_{N-1})$ , which is the Verlinde algebra at level  $k$ .

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